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Mixing Properties and the Chromatic Number of Ramanujan Complexes

Shai Evra, Konstantin Golubev, and Alexander Lubotzky

Einstein Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel

Dedicated to Nati Linial on his 60th birthday

Correspondence to be sent to: alex.lubotzky@math.huji.ac.il

Ramanujan complexes are high-dimensional simplicial complexes generalizing Ramanujan graphs. A result of Oh on quantitative property (T) for Lie groups over local fields is used to deduce a Mixing Lemma for such complexes. As an application, we prove that nonpartite Ramanujan complexes have "high girth" and high-chromatic number, generalizing a well-known result about Ramanujan graphs.

1 Introduction

In 1959, Erdős [4] used random methods to show that there are graphs with arbitrary large girth and arbitrary large chromatic number. In a way this is a surprising fact, since large girth means that such a graph looks locally like a tree, and so locally its chromatic number is two, while globally it requires a large number of colors. A constructive proof was given by Lovász [10] in 1968, and explicit examples (with quantitative estimates) in 1988 by Lubotzky *et al.* [14] using Ramanujan graphs. This is still by no mean an easy result even by nowadays standards.

The goal of this paper is to extend the above from Ramanujan graphs to high-dimensional Ramanujan complexes, as defined and constructed in [15, 16] (see also

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[9, 19]). One is facing the immediate question what we mean by "girth" and "chromatic number" for simplicial complexes?

The girth $g(X)$ of a graph X is equal to twice its injectivity radius $r(X)$ (more precisely $r(X) = \lfloor \frac{g(X)-1}{2} \rfloor$). The injectivity radius of X is the maximal $r \in \mathbb{N}$ such that if $\pi : \tilde{X} \to X$ is the universal cover map, then for every $y \in \tilde{X}$, π is one-to-one on the ball of radius r around y. This definition is easily extended to finite simplicial complexes and, in particular, to the Ramanujan complexes, whose universal covers are the Bruhat–Tits buildings of type $\tilde{A}$. The injectivity radius is defined similarly with respect to the graph metric on the 1-skeletons of $\tilde{X}$ and X. This notion has been studied in [13] where it was shown that there exist Ramanujan complexes of "large girth" in this sense. See Proposition 3.3 there and Corollary 5.2 below.

Before moving to the chromatic number, let us make the following warning: every simplicial complex X can be considered as a hypergraph H, when we take the maximal simplices (facets) of X to be the edges of H. Moreover if X is pure, that is, all its facets are of the same dimension, say $d-1$, then H is a d-uniform hypergraph, that is, all of its edges are of size d. The commonly used notion of girth in the theory of hypergraphs is different than the one we are using here; it refers to the length of a minimal sequence of the form $x_1, E_1, x_2, E_2, \ldots, x_g, E_g, x_{g+1}$ where all $x_1, \ldots, x_g$ are different vertices, $x_{g+1} = x_1$, $E_1, \ldots, E_g$ are edges and for any $i = 1, \ldots, g$, $\{x_i, x_{i+1}\} \subset E_i$. This notion is not suitable for the Ramanujan complexes or any clique complex: any two facets with a common 1-codimension wall give girth 2 in this definition (so, even the building has girth 2). Anyway, the theory of hypergraphs of high girth and high-chromatic number has been developed quite intensively. See [17] for a nice survey. The reader is referred also to [6, 7, 13] for related notions of girth for simplicial complexes, based on local acyclicity.

The notion of chromatic number for simplicial complexes we will use is the same as the one commonly used for hypergraphs. Let X be a $(d-1)$-dimensional simplicial complex with a set of vertices V.

Definition 1.1. The chromatic number of X, denoted $\chi(X)$, is the minimal number of colors needed to color the vertices of X, so that no facet (i.e., maximal face) is monochromatic. □

Clearly $\chi(X)$ is bounded from above by the chromatic number of the graph $X^{(1)}$(= the 1-skeleton of X).

Let us now recall what are Ramanujan complexes and how they are constructed: Let F be the local nonarchimedean field $\mathbb{F}_q((t))$, that is, the field of Laurent power series over $\mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of order q. Let $\mathcal{B}=\mathcal{B}_d(F)$ be the Bruhat–Tits building associated with $\mathrm{PGL}_d(F)$. It is an infinite $(d-1)$-dimensional countable simplicial complex, whose vertices come naturally with types in $\mathbb{Z}/d\mathbb{Z}$, denoted $\tau:\mathcal{B}(0)\to\mathbb{Z}/d\mathbb{Z}$ (see [15] and the references therein) in such a way that in every $(d-1)$-face all vertices are of different types. In particular, its chromatic number is at most d. (Even its 1-skeleton has chromatic number d.) In fact, its chromatic number is 2, since we can divide the set of d types $\mathbb{Z}/d\mathbb{Z}$ into two nonempty disjoint sets and then using only two colors, we get that no $(d-1)$-cell is monochromatic. If Γ is a co-compact lattice in $G=\mathrm{PGL}_d(F)$, with $\mathrm{dist}(\Gamma):=\min_{1\neq\gamma\in\Gamma,x\in\mathcal{B}}\mathrm{dist}(\gamma.x,x)\geq 2$, then $\Gamma\backslash\mathcal{B}$ is a finite simplicial complex. If Γ preserves the types of the vertices of $\mathcal{B}$, $\Gamma\backslash\mathcal{B}$ is also d-colorable (and even 2-colorable). The injectivity radius of $\Gamma\backslash\mathcal{B}$ is $\lfloor\frac{\mathrm{dist}(\Gamma)-1}{2}\rfloor$.

We will use the remarkable lattice Λ constructed by Cartwright and Steger [2], which acts transitively on the vertices of $\mathcal{B}$, and, in particular, does not preserve the types of the vertices of $\mathcal{B}$. In this Λ we will choose suitable sequence of congruence subgroups $\Lambda(f_n)$ for some $f_n\in\mathbb{F}_q[\frac{1}{t}]$, $n\in\mathbb{N}$, and show that the sequence of simplicial complexes, $X_n=\Lambda(f_n)\backslash\mathcal{B}$, satisfies

Theorem 1.2. For every integer $d\geq 3$ and odd prime power q such that $(q,d)=1$, there exists a sequence of finite $(d-1)$-dimensional simplicial complexes $(X_n)_{n\in\mathbb{N}}$ with $|X_n|\to\infty$, covered by $\mathcal{B}_d(\mathbb{F}_q((t)))$, with injectivity radius

$$r(X_n)\geq\frac{\log_q|X_n|}{2(d-1)(d^2-1)}-\frac{1}{2}$$

(so, the chromatic number of every ball of radius $\frac{\log_q|X_n|}{2(d-1)(d^2-1)}-\frac{1}{2}$ is two), while

$$\chi(X_n)\geq\tfrac{1}{2}\cdot q^{\frac{1}{2d}}$$

and so, $\chi(X_n)\to\infty$ when $q\to\infty$. In particular, by letting $q\to\infty$, this gives for every $d\geq 3$, $(d-1)$-dimensional simplicial complexes of arbitrarily large "girth" (twice the injectivity radius) and arbitrarily large chromatic number. □

Note that in order to have arbitrarily large chromatic number, q must go to infinity, otherwise the chromatic number of X_n, even as graphs, would be bounded since the degree would be bounded.

Moreover, for these complexes

$$\mathrm{diam}(X_n)\leq\frac{4\log_q|X_n|}{d^2}\leq 8d\cdot r(X_n),$$

for $q \gg d$ (see Remark 6.2). In particular, up to radius $\frac{\mathrm{diam}(X_n)}{8d}$, the chromatic number of a ball in X_n is 2, and only for bigger balls it grows, eventually to infinity.

As mentioned before, the fact that the quotients by congruence subgroups give large injectivity radius (and no small nontrivial homology cycles) was shown by Lubotzky and Meshulam in [13] (see [8, Section 4.1] for a "general principle" of this kind). So the main novelty of the current paper is giving a lower bound on the chromatic number for some carefully chosen congruence subgroups (see Section 5.3 below). To this end, we will prove the following result which is of independent interest.

Theorem 1.3 (Colorful Mixing Lemma). Let F be a nonarchimedean local field with finite residue field $\mathbb{F}_q$, q odd, $d \geq 3$, and $\mathcal{B} = \mathcal{B}_d(F)$, the Bruhat–Tits building associated with $\mathrm{PGL}_d(F)$. Let $\Gamma \leq \mathrm{PGL}_d(F)$ be a co-compact lattice preserving the type (coloring) function of $\mathcal{B}(0)$ with injectivity radius ≥ 2, so $X = \Gamma \backslash \mathcal{B}$ is a simplicial complex with a type function $\tau : X(0) \to \mathbb{Z}/d\mathbb{Z}$. For each type $i \in \mathbb{Z}/d\mathbb{Z} = \{1, 2, \ldots, d\}$, let $V_i \subset X(0)$ be the set of vertices of type i, that is, $V_i = \tau^{-1}(i)$.

Then for any choice of subsets $W_i \subseteq V_i$ we have

$$\left| \frac{|E(W_1, \ldots, W_d)|}{|X(d-1)|} - \prod_{i=1}^{d} \frac{|W_i|}{|V_i|} \right| \leq \frac{2d}{q^{\frac{1}{2}}},$$

where $E(W_1, \ldots, W_d)$ is the set of all $(d-1)$-dimensional cells with exactly one vertex in each W_i, $i = 1, \ldots, d$. □

So, the lemma ensures that when $q \gg 0$, the number $|E(W_1, \ldots, W_d)|$ of facets with one vertex from each W_i is approximately what one should expect by random considerations.

This mixing lemma will be deduced from a more general one (see Corollary 3.7 below) using a result of Oh [18] which gives a quantitative estimate for Kazhdan property (T) of $\mathrm{PGL}_d(F)$. In this argument, we follow a related use of Oh's work in [5].

It is interesting to observe that the above mixing lemma is for quotients of $\mathcal{B} = \mathcal{B}_d(F)$ on which the d-coloring by the d types is preserved, but eventually our main theorem is about $X_f = \Lambda(f) \backslash \mathcal{B}$ which are not d-colorable (in fact, our main goal is to show that they need many more colors!) We will acquire this by applying the colorful mixing lemma to the natural d-colorable d-sheeted cover of X_f (see Section 5 for details).

The paper is organized as follows. After a few preliminaries in Section 2, we show in Section 3 how the discrepancy of a colorful simplicial complex can be estimated using the eigenvalues of some naturally associated bipartite graphs. In Section 4, we will use Oh's theorem and apply it to the colorable quotients of the Bruhat–Tits building of $\mathrm{PGL}_d(F)$, to estimate these eigenvalues. In Section 5, we will follow carefully [16] to choose the suitable congruence subgroups $\Lambda(f)$ of Λ—the Cartwright–Steger lattice. We will use the congruence subgroups $\Lambda(f)$ for which $\Lambda(f)\backslash\mathcal{B}$ is a nonpartite complex, see there. In Section 6, we collect all the information together and prove Theorem 1.2.

This paper is dedicated to Nati Linial who has pioneered the study of high-dimensional expanders and many other things.

2 Notations and Conventions

Throughout this paper H is a finite d-uniform hypergraph, that is, $H=(V,E)$ and $E\subset\binom{V}{d}$. We say that H has a d-type function $\tau: V=V(H)\to\mathbb{Z}/d\mathbb{Z}$ if each edge contains vertices of all d types, that is, τ is one-to-one when restricted to each edge $e\in E$. We call such hypergraph d-partite and we also write it as $H=(V_0,\ldots,V_{d-1},E)$, where $V_i=\tau^{-1}(i), i\in\mathbb{Z}/d\mathbb{Z}$, and so E can be considered as a subset of $\prod_{i=0}^{d-1}V_i$. A 2-partite hypergraph is what is usually called a bipartite graph. Sometimes it is more convenient to think of $\mathbb{Z}/d\mathbb{Z}$ as $\{1,\ldots,d\}$.

Recall that a simplicial complex $X=(V,E)$ is a family E of finite subsets (called faces or simplicies) of the set of vertices V closed under inclusion, that is, if $F_1\in E$ and $F_2\subseteq F_1$ then $F_2\in E$.

For $F\in E$, denote $\dim(F)=|F|-1$ and $X(i)$ the set of simplices of dimension i. We say that $\dim(X)=d$ if $X(d)\neq\emptyset$ while $X(d+1)=\emptyset$. It is called a pure complex of dimension d, if every maximal face in E is of dimension d. Given $X=(V,E)$, we denote by $X^{(i)}$ the i-skeleton of X, this is the subcomplex of X of all the faces F in E with $\dim(F)\leq i$.

Given a pure simplicial complex $X=(V,E)$ of dimension $(d-1)$, one can associate with it the d-uniform hypergraph $H=\tilde{H}(X)=(V,X(d-1))$. Conversely if $H=(V,E)$ is a d-uniform hypergraph then by taking $\tilde{E}=\{F\subseteq V\mid\exists e\in E \text{ with } F\subseteq e\}$ we get a pure simplicial complex $X=\tilde{X}(H)=(V,\tilde{E})$ of dimension $d-1$. Clearly, $\tilde{X}(\tilde{H}(X))=X$ and $\tilde{H}(\tilde{X}(H))=H$. Moreover if τ is a type function on H, it defines a type function on X such that when restricted to every maximal face (facet) it is one-to-one. Such complexes are called balanced.

The theories of pure simplicial complexes and uniform hypergraphs are therefore completely equivalent. In this paper, we will use these languages interchangeably.

3 Discrepancy

For a d-partite hypergraph $H=(V_1,\ldots,V_d,E)$, and a collection of subsets $W_i\subseteq V_i$, $i=1,\ldots,d$, denote $E(W_1,\ldots,W_d)=E\cap\prod_{i=1}^d W_i$, the edges in E with vertices in $W_1,\ldots,W_d$. We define the *discrepancy* of $W_1,\ldots,W_d$ in H to be

$$\mathrm{disc}_H(W_1,\ldots,W_d)=\left|\frac{|E(W_1,\ldots,W_d)|}{|E|}-\prod_{i=1}^d\frac{|W_i|}{|V_i|}\right|.$$

In other words, the discrepancy measures the difference between the actual portion of edges between $W_1,\ldots,W_d$ and the expected portion if the hyperedges would have been chosen randomly uniformly.

For a biregular bipartite graph, the expander mixing lemma provides an upper bound on the discrepancy in the terms of the the second largest eigenvalue of the graph. In this section, our aim is to give a similar bound for d-partite hypergraphs.

3.1 Discrepancy of bipartite graphs

Let $G=(V_1,V_2,E)$ be a finite connected bipartite (k_1,k_2)-biregular graph on n vertices, that is, each vertex in V_1 has exactly k_1 neighbors, all of them in V_2, and each vertex in V_2 has k_2 neighbors, all of them in V_1, $|V_1|+|V_2|=n$, and so $k_1|V_1|=k_2|V_2|=|E|$.

Recall that the adjacency operator $A=A(G)$ of the graph G is the following operator on the space of complex valued functions on the vertices

$$(Af)(v)=\sum_{u\sim v}f(u),$$

where $u\sim v$ stands for $(u,v)\in E$.

The following lemmas are probably known, but for lack of a suitable reference we give short proofs.

Lemma 3.1. Let $\lambda_n\le\cdots\le\lambda_2\le\lambda_1$ be the eigenvalues of the adjacency operator A of G. Then

(1) The spectrum is symmetric, that is, $\lambda_{n-i+1}=-\lambda_i$ for all i.
(2) The largest (respectively, smallest) eigenvalue is $\lambda_1=\sqrt{k_1k_2}$ (respectively, $\lambda_n=-\sqrt{k_1k_2}$), whose corresponding eigenfunction is $\sqrt{k_1}\mathbb{1}_{V_1}+\sqrt{k_2}\mathbb{1}_{V_2}$ (respectively, $\sqrt{k_1}\mathbb{1}_{V_1}-\sqrt{k_2}\mathbb{1}_{V_2}$). □

Proof. Let f be an eigenfunction of A with an eigenvalue λ, that is, $Af=\lambda\cdot f$. Then it is easy to see that the following function

$$g(v)=\begin{cases} f(v) & \text{if } v\in V_1 \\ -f(v) & \text{if } v\in V_2 \end{cases}$$

satisfies $Ag=(-\lambda)\cdot g$, which proves (1).

If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2. The operator A^2 expresses the 2-step walk on G, that is, the (v,u)-entry in the matrix of A^2 equals the number of paths of length 2 in G connecting the vertices v and u. By the (k_1,k_2)-regularity condition, the sum of any row of the matrix A^2 is k_1k_2. Thus, A^2 is the adjacency matrix of a k_1k_2-regular multigraph, that is, a graph with loops and multiple edges, and hence its largest eigenvalue is k_1k_2. Thus the largest eigenvalue of A is $\sqrt{k_1k_2}$, and by (1), the smallest is $-\sqrt{k_1k_2}$.

By the biregularity condition,

$$A(\mathbb{1}_{V_1})=k_2\mathbb{1}_{V_2} \quad \text{and} \quad A(\mathbb{1}_{V_2})=k_1\mathbb{1}_{V_1}.$$

Hence

$$A(\sqrt{k_1}\mathbb{1}_{V_1}\pm\sqrt{k_2}\mathbb{1}_{V_2})=\pm\sqrt{k_1k_2}(\sqrt{k_1}\mathbb{1}_{V_1}\pm\sqrt{k_2}\mathbb{1}_{V_2}).$$ ■

Lemma 3.2 (Expander mixing lemma for bipartite graphs). Let $G=(V_1,V_2,E)$ be a bipartite (k_1,k_2)-biregular finite connected graph. Let $\lambda=\lambda(G)$ be the second largest eigenvalue of the adjacency operator of G. Then for any $S\subseteq V_1$ and $T\subseteq V_2$,

$$\left| |E(S,T)|-\frac{\sqrt{k_1k_2}|S||T|}{\sqrt{|V_1||V_2|}}\right| \le \lambda(G)\sqrt{|S||T|}. \tag{1}$$

□

Proof. As before, denote by A the adjacency matrix of G, and its spectrum by $\lambda_n\le\cdots\le\lambda_2\le\lambda_1$. Note that

$$|\lambda_1|=|\lambda_n|=\sqrt{k_1k_2} \quad \text{and} \quad |\lambda_i|\le\lambda(G) \text{ for } i=2,\ldots,n-1.$$

Let $f_1,\ldots,f_n$ be an orthonormal basis of eigenfunctions of A, that is, $Af_i=\lambda_i f_i$ and $\langle f_i,f_j\rangle=\delta_{i,j}$, where the inner product is defined as

$$\langle f,g\rangle=\sum_{v\in V_1\sqcup V_2} f(v)\overline{g(v)}.$$

Let $\mathbb{1}_S$ and $\mathbb{1}_T$ be the characteristic functions of S and T, respectively. Then $|E(S,T)| = \langle A\mathbb{1}_S, \mathbb{1}_T\rangle$. Expressing them as linear combinations of orthonormal eigenvectors of A

$$\mathbb{1}_S = \sum_{i=1}^{n} s_i f_i \quad \text{and} \quad \mathbb{1}_T = \sum_{i=1}^{n} t_i f_i,$$

we get

$$|E(S,T)| = \langle A\mathbb{1}_S, \mathbb{1}_T\rangle = \sum_{i=1}^{n} s_i \bar{t}_i \lambda_i = s_1 \bar{t}_1 \lambda_1 + s_n \bar{t}_n \lambda_n + \sum_{i=2}^{n-1} s_i \bar{t}_i \lambda_i.$$

By Lemma 3.1, we can assume that

$$f_1 = \frac{1}{\sqrt{k_1|V_1| + k_2|V_2|}} (\sqrt{k_1}\mathbb{1}_{V_1} + \sqrt{k_2}\mathbb{1}_{V_2})$$

and

$$f_n = \frac{1}{\sqrt{k_1|V_1| + k_2|V_2|}} (\sqrt{k_1}\mathbb{1}_{V_1} - \sqrt{k_2}\mathbb{1}_{V_2}).$$

Hence, as $k_1|V_1| = k_2|V_2|$, we get

$$s_1 = \langle f_1, \mathbb{1}_S\rangle = \frac{|S|}{\sqrt{2|V_1|}} = s_n \quad \text{and} \quad \bar{t}_1 = \langle f_1, \mathbb{1}_T\rangle = \frac{|T|}{\sqrt{2|V_2|}} = -\bar{t}_n.$$

Therefore, since $\lambda_1 = \sqrt{k_1 k_2} = -\lambda_n$,

$$s_1 \bar{t}_1 \lambda_1 = s_n \bar{t}_n \lambda_n = \frac{\sqrt{k_1 k_2}|S||T|}{2\sqrt{|V_1||V_2|}}.$$

And so,

$$\left| |E(S,T)| - \frac{\sqrt{k_1 k_2}|S||T|}{\sqrt{|V_1||V_2|}} \right| = \left| \sum_{i=2}^{n-1} s_i \bar{t}_i \lambda_i \right| \le \lambda(G) \sum_{i=2}^{n-1} \left| s_i \bar{t}_i \right| \le \lambda(G) \sqrt{\left(\sum_{i=2}^{n-1} |s_i|^2 \right) \left(\sum_{i=2}^{n-1} |t_i|^2 \right)}$$

$$\le \lambda(G) \sqrt{\|\mathbb{1}_S\|^2 \|\mathbb{1}_T\|^2} \le \lambda(G) \sqrt{|S||T|}. \qquad \blacksquare$$

We learned recently that Lemma 3.2 appears also in [1].

Recall that the normalized adjacency operator $\tilde{A} = \tilde{A}(G)$ of a graph G is the following operator on the space of complex valued functions on the vertices

$$(\tilde{A} f)(v) = \frac{1}{\deg v} \sum_{u \sim v} f(u), \tag{2}$$

where $u \sim v$ stands for $(u,v) \in E$.

For a biregular bipartite graph G if f is an eigenfunction of the adjacency operator $A(G)$ with an eigenvalue λ, then $\frac{1}{\sqrt{\deg v}} f(v)$ is an eigenfunction of the normalized adjacency operator $\tilde{A}(G)$ with an eigenvalue $\frac{\lambda}{\sqrt{k_1 k_2}}$.

In particular, the largest and smallest eigenvalues of the normalized adjacency operator are 1 and (-1), respectively. The second largest eigenvalue $\tilde{\lambda}$ of $\tilde{A}$ is equal to

$$\tilde{\lambda}(G) = \frac{\lambda(G)}{\sqrt{k_1 k_2}}. \tag{3}$$

Definition 3.3. Let $G = (V_1, V_2, E)$ be a bipartite graph, $S \subseteq V_1$ and $T \subseteq V_2$. Then the discrepancy of these subsets is defined to be

$$\mathrm{disc}_G(S, T) = \left| \frac{|E(S, T)|}{|E|} - \frac{|S|}{|V_1|} \frac{|T|}{|V_2|} \right|. \qquad \square$$

In these terms, the following statement is a corollary of the Expander Mixing Lemma (Lemma 3.2).

Corollary 3.4. Let $G = (V_1, V_2, E)$ be a bipartite (k_1, k_2)-biregular finite graph. Then for any $S \subseteq V_1$, $T \subseteq V_2$

$$\mathrm{disc}_G(S, T) \leq \tilde{\lambda}(G) \cdot \sqrt{\frac{|S|}{|V_1|} \frac{|T|}{|V_2|}} \qquad \square$$

Proof. By the (k_1, k_2)-biregularity of G

$$|E| = k_1 |V_1| = k_2 |V_2| = \sqrt{k_1 k_2 |V_1||V_2|},$$

hence

$$\frac{\sqrt{k_1 k_2} |S||T|}{\sqrt{|V_1||V_2|}} = |E| \cdot \left(\frac{|S||T|}{|V_1||V_2|} \right).$$

And therefore, by Lemma 3.2, we get

$$\left| \frac{|E(S, T)|}{|E|} - \frac{|S|}{|V_2|} \frac{|T|}{|V_1|} \right| \leq \frac{\lambda(G)}{\sqrt{k_1 k_2}} \sqrt{\frac{|S||T|}{|V_1||V_2|}}. \qquad \blacksquare$$

3.2 Discrepancy of hypergraphs

Let $H = (V_1, \ldots, V_d, E)$ be a d-partite hypergraph. Our aim is to give estimates and bounds on its discrepancy. We will do it by defining various associated bipartite graphs and then we will bound the discrepancies of H by their discrepancies.

For $i = 1, \ldots, d$, denote $E_i = \{F \setminus \{v_i\} \mid F \in E, v_i \in V_i\}$, that is, the set E_i is the set of of all edges of H with the vertex of type i being removed. A set $Y \in E_i$ is called a *wall* *of cotype* i. Denote by $H_i = (V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_d, E_i)$ the $(d-1)$-partite hypergraph induced from H by removing the vertices of type i.

Denote by B_i the bipartite graph with V_i as one set of vertices and E_i as the second. A vertex $v_i \in V_i$ and a wall $Y \in E_i$ are connected by an edge of B_i if their union forms an edge of H, that is, $\{v_i\} \cup Y \in E$. We will write $B_i = (V_i, E_i, E_{B_i})$. An edge of B_i is a pair $(v_i, F \setminus \{v_i\})$, where $F \in E$ is an edge of H. Following the terminology of simplicial complexes, this is the "vertices versus walls" graph. Note that since every edge of H has exactly one vertex in V_i, there is a natural bijection between the edges of B_i and the edges of H.

As before, for a collection of subsets $W_j \subseteq V_j$ for $j = 1, \ldots, d$, we denote by $E(W_1, \ldots, W_d)$ the set of all edges of H with vertices in the sets $W_1, \ldots, W_d$. Analogously, for H_i we will denote by $E_i(W_1, \ldots, W_d)$ the subset of all edges with vertices in $W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_d$. For the graph B_i we will denote by $E_{B_i}(W_i, E_i(W_1, \ldots, W_d))$ the set of all edges of B_i with one vertex in W_i and the other in $E_i(W_1, \ldots, W_d)$. Note that the above-mentioned bijection between the edges of H and B_i restricts to a bijection between $E(W_1, \ldots, W_d)$ and $E_{B_i}(W_i, E_i(W_1, \ldots, W_d))$.

The following lemma reduces the question of bounding the discrepancy of a d-partite hypergraph to its induced hypergraphs and bipartite graphs.

Lemma 3.5. Let $W_j \subseteq V_j$ for $j = 1, \ldots, d$. Then for $i = 1, \ldots, d$,

$$\mathrm{disc}_H(W_1, \ldots, W_d) \le \mathrm{disc}_{B_i}(W_i, E_i(W_1, \ldots, W_d)) + \frac{|W_i|}{|V_i|}\mathrm{disc}_{H_i}(W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_d). \tag{4}$$

□

Proof. By the definition of the bipartite graph B_i

$$\frac{|E(W_1, \ldots, W_d)|}{|E|} = \frac{|E_{B_i}(W_i, E_i(W_1, \ldots, W_d))|}{|E_{B_i}|}$$

and hence

$$\begin{aligned} &\mathrm{disc}_H(W_1, \ldots, W_d) \\ &= \left| \frac{|E(W_1, \ldots, W_d)|}{|E|} - \prod_{j=1}^{d} \frac{|W_j|}{|V_j|} \right| \\ &= \left| \frac{|E(W_1, \ldots, W_d)|}{|E|} - \frac{|W_i|}{|V_i|} \cdot \frac{|E_i(W_1, \ldots, W_d)|}{|E_i|} + \frac{|W_i|}{|V_i|} \cdot \left(\frac{|E_i(W_1, \ldots, W_d)|}{|E_i|} - \prod_{j=1, j\neq i}^{d} \frac{|W_j|}{|V_j|} \right) \right| \end{aligned}$$

$$\le \left| \frac{|E_{B_i}(W_1, \ldots, W_d)|}{|E_{B_i}|} - \frac{|W_i|}{|V_i|} \cdot \frac{|E_i(W_1, \ldots, W_d)|}{|E_i|} \right| + \frac{|W_i|}{|V_i|} \cdot \left| \frac{|E_i(W_1, \ldots, W_d)|}{|E_i|} - \prod_{j=1, j\neq i}^{d} \frac{|W_j|}{|V_j|} \right|$$

$$= \mathrm{disc}_{B_i}(W_i, E_i(W_1, \ldots, W_d)) + \frac{|W_i|}{|V_i|} \cdot \mathrm{disc}_{H_i}(W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_d). \qquad \blacksquare$$

Definition 3.6. A d-partite hypergraph H is called type-regular if for any type i, $1 \le i \le d$, there exist $k_i, l_i \in \mathbb{N}$, such that each i-type vertex is contained in exactly k_i hyperedges in H and each cotype i wall is contained in exactly l_i hyperedges in H. Note that if H is type-regular, then each induced bipartite graph B_i, defined above, is (k_i, l_i)-biregular. □

Recall that for a graph G we denote by $\tilde{\lambda}(G)$ the normalized second largest eigenvalue. We can now generalize Corollary 3.4 from graphs to hypergraphs.

Corollary 3.7. Let H be a d-partite type-regular hypergraph. Let $B_i = (V_i, E_i, E_{B_i})$, $i = 1 \ldots, d$, be the induced bipartite graphs of H, as defined above. Then for any $W_1 \subseteq V_1, \ldots, W_d \subseteq V_d$,

$$\mathrm{disc}_H(W_1, \ldots, W_d) \le \sum_{i=1}^{d-1} \left(\tilde{\lambda}(B_i) \cdot \sqrt{\frac{|W_i|}{|V_i|}} \right).$$

In particular

$$\max_{W_i \subseteq V_i} \mathrm{disc}_H(W_1, \ldots, W_d) \le (d-1) \cdot \max_{1 \le i \le d-1} \tilde{\lambda}(B_i).$$

□

Proof. Since H is type-regular, for any type $i \in \mathbb{Z}/d\mathbb{Z}$, there exists a number $l_i \in \mathbb{N}$, such that any wall of cotype i is contained in exactly l_i facets. Hence,

$$l_i |E_i(W_1, \ldots, W_d)| = |E(W_1, \ldots, W_{i-1}, V_i, W_{i+1}, \ldots, W_d)| \quad \text{and} \quad l_i |E_i| = |E|,$$

and so,

$$\mathrm{disc}_{H_i}(W_1, \ldots, W_d) = \left| \frac{|E_i(W_1, \ldots, W_d)|}{|E_i|} - \prod_{j=1, j\neq i}^{d} \frac{|W_j|}{|V_j|} \right|$$

$$= \left| \frac{|E(W_1, \ldots, W_{i-1}, V_i, W_{i+1}, \ldots, W_d)|}{|E|} - \frac{|V_i|}{|V_i|} \prod_{j=1, j\neq i}^{d} \frac{|W_j|}{|V_j|} \right|$$

$$= \mathrm{disc}_H(W_1, \ldots, W_{i-1}, V_i, W_{i+1}, \ldots, W_d). \tag{5}$$

Corollary 3.4 gives that for any $i = 1, \ldots, d$,

$$\mathrm{disc}_{B_i}(W_i, E_i(V_1, \ldots, V_{i-1}, W_{i+1}, \ldots, W_d)) \leq \tilde{\lambda}(B_i) \cdot \sqrt{\frac{|W_i|}{|V_i|}}.$$

So, by iterating on Lemma 3.5 and Equation (5), we get

$$\begin{aligned}\mathrm{disc}_H(W_1, \ldots, W_d) &\leq \tilde{\lambda}(B_1) \cdot \sqrt{\frac{|W_1|}{|V_1|}} + \mathrm{disc}_H(V_1, W_2, \ldots, W_d) \\ &\leq \tilde{\lambda}(B_1) \cdot \sqrt{\frac{|W_1|}{|V_1|}} + \tilde{\lambda}(B_2) \cdot \sqrt{\frac{|W_2|}{|V_2|}} + \mathrm{disc}_H(V_1, V_2, W_3, \ldots, W_d) \\ &\leq \cdots \leq \sum_{i=1}^{d-1} \tilde{\lambda}(B_i) \cdot \sqrt{\frac{|W_i|}{|V_i|}} + \mathrm{disc}_H(V_1, V_2, \ldots, V_{d-1}, W_d) = \sum_{i=1}^{d-1} \tilde{\lambda}(B_i) \cdot \sqrt{\frac{|W_i|}{|V_i|}}.\end{aligned}$$

The last equality follows from the fact that $\mathrm{disc}_H(V_1, V_2, \ldots, V_{d-1}, W_d) = 0$, since any vertex w of W_d is contained in k_d edges and for elements $w \neq w'$ of W_d these edges are different. ■

4 Colorful Mixing Lemma for Ramanujan Complexes

The goal of this section is to prove the Colorful Mixing Lemma (Theorem 1.3).

Let F be a local nonarchimedean field whose residue field is of order q, and let $\mathcal{B} = \mathcal{B}_d(F)$, $d \geq 3$, be the Bruhat–Tits building of type $\tilde{A}_{d-1}$ associated with $\mathrm{PGL}_d(F)$. The building is equipped with a natural d-type function which gives it a structure of an infinite d-partite hypergraph. For any co-compact lattice $\Gamma \leq \mathrm{PGL}_d(F)$ preserving the type function, the quotient $\mathcal{B}_\Gamma = \Gamma \backslash \mathcal{B}_d(F)$ is a finite d-partite hypergraph. Recall the notation $\mathrm{dist}(\Gamma) = \min_{x \in \mathcal{B}, 1 \neq \gamma \in \Gamma} \mathrm{dist}(\gamma.x, x)$, and that the injectivity radius $r(\Gamma)$ of $\mathcal{B}_\Gamma$ is equal to $\lfloor \frac{\mathrm{dist}(\Gamma)-1}{2} \rfloor$, since the building $\mathcal{B}$ is its universal cover. The Colorful Mixing Lemma reads as follows. Assuming that the injectivity radius of $\mathcal{B}_\Gamma$ is at least 2, for any choice of subsets W_i of vertices of $\mathcal{B}_\Gamma$ of type i

$$\mathrm{disc}_{\mathcal{B}_\Gamma}(W_1, \ldots, W_d) \leq \frac{2d}{q^{1/2}}. \tag{6}$$

4.1 The building of type $\tilde{A}_{d-1}$

In this section, we review the structure and basic properties of the building $\mathcal{B}_d(F)$. Rather than using the general language of buildings, we will present it and prove its properties from basic principles.

Let F be a local nonarchimedean field with a discrete valuation $\nu : F^* \to \mathbb{Z}$, let $\mathcal{O}$ be the ring of integers of F, π a uniformizer and $q < \infty$ the cardinality of the residue field $\bar{F} = \mathcal{O}/\pi\mathcal{O}$. For example, $F = \mathbb{F}_q((t))$ the field of Laurent series over $\mathbb{F}_q$, $\nu = \deg$, $\mathcal{O} = \mathbb{F}_q[\![t]\!]$ the Taylor series and $\pi = t$.

The building $\mathcal{B} = \mathcal{B}_d(F)$ associated with a local field F is an infinite $(d-1)$-dimensional pure simplicial complex constructed as follows.

Vertices. A *lattice* is a free $\mathcal{O}$-submodule of $V = F^d$ of rank d, that is, it is of the form $\langle v_1, \ldots, v_d \rangle = \mathcal{O}v_1 + \cdots + \mathcal{O}v_d$ where $\{v_1, \ldots, v_d\}$ is a basis for V. Two lattices L_1, L_2 are said to be equivalent if there exists $\lambda \in F^*$ such that $L_1 = \lambda L_2$. The equivalence class of a lattice L is denoted by $[L]$. The set of equivalence classes of lattices forms the set of vertices of the building $\mathcal{B}$.

Faces. Two vertices $[L_1], [L_2]$ are connected by an edge in the building if there exist representatives $L'_1 \in [L_1], L'_2 \in [L_2]$ such that $\pi L'_1 \subset L'_2 \subset L'_1$. Note that $L'_1/\pi L'_1$ is a d-dimensional vector space over the finite field $\bar{F} = \mathcal{O}/\pi\mathcal{O}$. Fixing a representative L'_1 of a vertex $[L_1]$ gives rise to a one-to-one correspondence between the neighbors of $[L_1]$ and the proper subspaces of $L'_1/\pi L'_1$.

A set of vertices $\{[L_1], \ldots, [L_k]\}$ forms a $(k-1)$-face in the building if there exist representatives $L'_i \in [L_i]$ such that (maybe, after renumbering) $\pi L'_1 \subset L'_k \subset \cdots \subset L'_2 \subset L'_1$. Note that a $(k-1)$-simplex in the building gives rise to a k-flag of subspaces in $L'_1/\pi L'_1$, hence the dimension of the building is $(d-1)$.

The link of every vertex of $\mathcal{B}$ is isomorphic to the flag complex of $\bar{F}^d = \mathbb{F}_q^d$.

Action of $GL_d(F)$. The group $GL_d(F)$ acts transitively on the lattices in F^d, and its center preserves the equivalence classes, hence this action induces an action of $G = \mathrm{PGL}_d(F)$ on the vertices of the building. The stabilizer of the vertex $[\mathcal{O}^d]$, which is called *the standard lattice*, is $K = \mathrm{PGL}_d(\mathcal{O})$, hence the set G/K may be identified with the set of vertices of the building, and multiplication on the left by G on G/K may be identified with the action of G on the building.

Type function. For each vertex $[L]$ there exists an element $g \in G$ such that $[L] = g[\mathcal{O}^d]$. Define the type of $[L]$ to be $\tau([L]) = \nu(\det(g)) \bmod d$. It is well defined since for $k \in K$ the determinant $\det(k) \in \mathcal{O}^*$ and hence $\nu(\det(k)) = 0$. This defines a type function from the vertices of the building to $\mathbb{Z}/d\mathbb{Z}$. Note that a maximal simplex contains vertices of all d types.

For $i \in \mathbb{Z}/d\mathbb{Z}$, denote $G_i = (\nu \circ \det)^{-1}(i)$. Then G_0 is the subgroup of G of type-preserving elements, and G_i are its cosets. We saw before that the vertices of the building may be identified with G/K. Under this identification, G_i/K is the set of vertices of type i.

The group G_0 of type-preserving elements is equal to $\mathrm{PSL}_d(F)\cdot K$. It is a normal subgroup of $G=\mathrm{PGL}_d(F)$ of index d, since $G_0=\{g\in G \mid \nu(\det(g))=0 \bmod d\}$.

Let now Γ be a co-compact discrete subgroup of G which is contained in G_0 and $\mathcal{B}_\Gamma=\Gamma\backslash\mathcal{B}$. This is a finite simplicial complex and τ is well defined on $\mathcal{B}_\Gamma$. The vertices V of $\mathcal{B}_\Gamma$ may be identified with $\Gamma\backslash G/K$, and in this case the i-typed vertices V_i of $\mathcal{B}_\Gamma$ are identified with $\Gamma\backslash G_i/K$.

Relative position. Define the set $A^+=\{a=\overline{(a_1,\ldots,a_d)}\in\mathbb{Z}^d/(1,\ldots,1)\mathbb{Z} \mid a_1\le\cdots\le a_d\}$, and let Λ^+ be the set of diagonal matrices in $\mathrm{PGL}_d(F)$ of the form $\pi^a=\pi^{\overline{(a_1,\ldots,a_d)}}=diag(\pi^{a_1},\ldots,\pi^{a_d})$ for $\overline{(a_1,\ldots,a_d)}\in A^+$. Denote $A_0=\{\overline{(a_1,\ldots,a_d)}\in A \mid \sum a_i\equiv 0 \bmod d\}$. The Cartan decomposition $G=K\Lambda^+K$, means that each element $g\in G$ may be written uniquely as $g=k_1\pi^a k_2$ for $k_1,k_2\in K$ and $a\in A^+$. By identifying the vertices of the building with G/K, for any two vertices $x=gK$ and $y=hK$ we define the *relative position* of y w.r.t. x, to be the unique element $a\in A^+$ such that $Kg^{-1}hK=K\pi^aK$. We get a function $\mathcal{B}(0)\times\mathcal{B}(0)\to A^+$, where $\mathcal{B}(0)$ is the set of vertices of $\mathcal{B}$.

In other words, for two vertices x and y consider a basis $\{v_1,\ldots,v_d\}$ of F^d, such that $x=[\mathcal{O}v_1+\cdots+\mathcal{O}v_d]$ and $y=[\pi^{a_1}\mathcal{O}v_1+\cdots+\pi^{a_d}\mathcal{O}v_d]$ (one can always find such a basis). Let $g\in G=\mathrm{PGL}_d(F)$ be the element which sends the standard basis to $\{v_1,\ldots,v_d\}$, and let $a=\overline{(a_1,\ldots,a_d)}\in A^+$. Then $x=g.[\mathcal{O}^d]$ and $y=(g\pi^a).[\mathcal{O}^d]$, hence "$x^{-1}y$" $=\pi^a$, and the relative position of y w.r.t. x is a. We also see that in this case $\tau(x)-\tau(y)\equiv-\sum a_i \bmod d$, and the relative position of x w.r.t. y is $\overline{(0,a_d-a_{d-1},\ldots,a_d-a_1)}$. In addition, if y is in relative position $\overline{(a_1,\ldots,a_d)}$ w.r.t. x, then the distance between them, that is, the number of edges in the shortest path connecting them, is equal to $\mathrm{dist}(x,y)=a_d-a_1$. The action of G on $\mathcal{B}$ preserves the relative position of pairs of vertices.

We note that by the Cartan decomposition, for any $a\in A^+$, K acts transitively on the vertices of a fixed relative position a w.r.t. the standard lattice $x_0=[\mathcal{O}^d]$. By the transitivity of the action of G, for any vertex x, $K_x=Stab_G(x)$ acts transitively on the vertices of relative position a w.r.t. x.

Various combinatorial aspects of the building can be expressed by the relative position.

Lemma 4.1. Let y be a vertex in the building with relative position $a=\overline{(a_1,\ldots,a_d)}$ w.r.t. x. Then

(1) x and y are neighbors if and only if $a_d=a_1+1$, that is, $a=\overline{(0,\ldots,0,1\ldots,1)}$.

(2) x and y are of the same type if and only if $\sum a_i=0 \bmod d$, that is, $a\in A_0$.

(3) x and y are separated by a common wall of codimension 1 (i.e., there exists a $(d-2)$-face σ such that $\sigma\cup\{x\}$ and $\sigma\cup\{y\}$ are both $(d-1)$-faces) if and

only if either $a = \overline{(0,\ldots,0)}$, that is, they coincide, or $a = \overline{(-1,0,\ldots,0,1)} = \overline{(0,1,\ldots,1,2)}$. □

Proof. Statements (1) and (2) follow immediately from the definition of the relative position and the discussion above.

To prove (3), assume first that y is in relative position $\overline{(0,1,\ldots,1,2)}$ w.r.t. x. This implies that x has a representative L with an $\mathcal{O}$-basis $\{v_1,\ldots,v_d\}$ such that $L' = \langle v_1, \pi v_2, \ldots, \pi v_{d-1}, \pi^2 v_d\rangle$ represents y. For $i = 1,\ldots,d-1$, denote $L_i = \langle v_1,\ldots,v_i,\pi v_{i+1},\ldots,\pi v_d\rangle$. Then the set $\sigma = \{z_i = [L_i] \mid 1 \le i \le d-1\}$ forms a $(d-2)$-cell, and both $\sigma \cup \{x\}$ and $\sigma \cup \{y\}$ are facets of $\mathcal{B}$, since

$$\pi L \subset L_1 \subset \cdots L_{d-1} \subset L \quad \text{and} \quad \pi L_{d-1} \subset L' \subset L_1 \subset \cdots \subset L_{d-1}.$$

To see the opposite direction, assume that there exists a $(d-2)$-cell σ with both $\sigma \cup \{x\}$ and $\sigma \cup \{y\}$ being facets of $\mathcal{B}$. As the link of every vertex of $\mathcal{B}$ is the flag complex of $\mathbb{F}_q^d$, one can deduce that σ is contained in $(q+1)$ facets. The fact that $\sigma \cup \{x\}$ is a facet, implies that there exists a representative L of x with an $\mathcal{O}$-basis $\{v_1,\ldots,v_d\}$ such that $\sigma = \{z_i = [L_i] \mid 1 \le i \le d-1\}$, where $L_i = \langle v_1,\ldots,v_i,\pi v_{i+1},\ldots,\pi v_d\rangle$.

Here is a list of representatives of $(q+1)$ vertices $\{y_\varepsilon = [L'_\varepsilon] \mid \varepsilon \in \mathbb{F}_q \cup \{\infty\}\}$ such that $\sigma \cup \{y_\varepsilon\}$ is a facet of $\mathcal{B}$

$$L'_\infty = \langle \pi v_1,\ldots,\pi v_d\rangle \quad (\text{note that } y_\infty = [L'_\infty] = x)$$

and for $\varepsilon \in \mathbb{F}_q$

$$L'_\varepsilon = \langle v_1 + \varepsilon \cdot \pi v_d, \pi v_2, \ldots, \pi v_{d-1}, \pi^2 v_d\rangle.$$

One can easily check that all the y_ε's are not equivalent and $\sigma \cup \{y_\varepsilon\}$ is a facet, so these are all the facets containing σ. For every $\varepsilon \in \mathbb{F}_q$, y_ε is in relative position $\overline{(0,1,\ldots,1,2)}$ w.r.t. x. This can be seen by taking $\{v_1 + \varepsilon \cdot \pi v_d, v_2, \ldots, v_d\}$ as a basis for L. ■

4.2 Hecke operators

For any $a = \overline{(a_1,\ldots,a_d)} \in A^+$, define the following Hecke operator on the vertices of the building $H_a : L^2(\mathcal{B}(0)) \to L^2(\mathcal{B}(0))$,

$$H_a f(xK) = \frac{1}{\mu(K\pi^a K)} \sum_{yK \in xK\pi^a K} f(yK),$$

where μ is the Haar measure on G, normalized such that $\mu(K) = 1$, (i.e., $\mu(K\pi^a K) = |K\pi^a K/K|$). This is the normalized finite sum over the vertices yK of relative position

a w.r.t. xK. Note that $\mu(K\pi^a K)$ is equal to the number of vertices which are of relative position a w.r.t. x.

A lattice Γ of G acts on the left while the Hecke operator H_a acts on the right, so these two actions on $L^2(G/K)$ commute. Hence, we can consider H_a also as a map $H_a : L^2(V) \to L^2(V)$ where V is the set of vertices of $\Gamma\backslash\mathcal{B}$, so

$$H_a f(\Gamma xK) = \frac{1}{\mu(K\pi^a K)} \sum_{\Gamma yK \in \Gamma xK\pi^a K} f(\Gamma yK).$$

Moreover if $a \in A_0$, then the type of each $yK \in xK\pi^a K$ is the same as that of xK, so we may consider H_a as a map $H_a : L^2(V_i) \to L^2(V_i)$, where V_i is the set of vertices of type i.

Finally, note that if we consider $f \in L^2(\Gamma\backslash G/K)$, as a right K-invariant function in $L^2(\Gamma\backslash G)$, and dk is the Haar measure on K, normalized such that $dk(K) = 1$, we may write H_a as an integral over K, instead of a sum,

$$H_a f(x) = \int_K f(xk\pi^a)\, \mathrm{d}k.$$

Let $(\rho, L^2(\Gamma\backslash G))$ be the unitary G-representation, given by right translations $\rho(g) f(x) = f(xg)$. The following lemma will allow us to give bounds on the spectra of the Hecke operators, assuming we have bounds on the matrix coefficients of the representation $L^2(\Gamma\backslash G)$. Bounds on the matrix coefficients will be given at the end of this section, using a theorem by Oh.

Lemma 4.2. Let $a \in A^+$. For any right K-invariant vectors f_1, f_2 in $L^2(\Gamma\backslash G)$,

$$\langle H_a f_1, f_2\rangle = \langle \rho(\pi^a) f_1, f_2\rangle$$

□

Proof.

$$\begin{aligned}\langle H_a f_1, f_2\rangle &= \int_{\Gamma\backslash G} H_a\,(f_1(x))\,\overline{f_2(x)}\,\mathrm{d}x \\ &= \int_{\Gamma\backslash G} \left(\int_K f_1(xk\pi^a)\,\mathrm{d}k\right) \overline{f_2(x)}\,\mathrm{d}x \\ &= \int_K \left(\int_{\Gamma\backslash G} f_1(xk\pi^a)\overline{f_2(x)}\,\mathrm{d}x\right) \mathrm{d}k,\end{aligned}$$

where the last equality follows from Fubini's theorem. Since the measure dx is right invariant we can replace x by xk, and by the right K-invariance of f_2 we get

$$\langle H_a f_1, f_2\rangle = \int_K \left(\int_{\Gamma\backslash G} f_1(x\pi^a)\overline{f_2(x)}\,\mathrm{d}x\right)\mathrm{d}k$$
$$= \int_{\Gamma\backslash G} f_1(x\pi^a)\overline{f_2(x)}\,\mathrm{d}x = \langle\rho(\pi^a) f_1, f_2\rangle. \qquad \blacksquare$$

Definition 4.3. Denote $G^+ = \mathrm{PSL}_d(F)$ and note that it is the subgroup of $G = \mathrm{PGL}_d(F)$ generated by unipotent elements of G. □

Definition 4.4. For a finite set X, denote $L^2_0(X) = \{g \in L^2(X) \mid \langle g, \mathbb{1}_X\rangle = 0\}$ the subspace of $L^2(X)$ of functions orthogonal to the constant function. □

Lemma 4.5. Let $i \in \mathbb{Z}/d\mathbb{Z}$. A function $f \in L^2_0(V_i)$ extended to a function on V by setting it zero outside V_i and regarded as a right K-invariant function in $L^2(\Gamma\backslash G)$ is orthogonal to any right G^+-invariant function in $L^2(\Gamma\backslash G)$. □

Proof. Let $h \in L^2(\Gamma\backslash G)$ be a right G^+-invariant function. Define $\tilde{h}(x) = \int_K h(xk)\,\mathrm{d}k$. Recall that the type-preserving subgroup G_0 is equal to G^+K. So, $\tilde{h}$ is right G^+-invariant and right K-invariant, and hence right G_0-invariant. Since each G_i is a G_0-coset, $\tilde{h}$ is constant on each $\Gamma\backslash G_i$ and hence on $V_i = \Gamma\backslash G_i/K$. Now, since $f \in L^2_0(V_i)$ we get that

$$\langle f, \tilde{h}\rangle = 0.$$

On the other hand,

$$\langle f, h\rangle = \int_{\Gamma\backslash G} f(x)\overline{h(x)}\,\mathrm{d}x = \int_K \left(\int_{\Gamma\backslash G} f(x)\overline{h(x)}\,\mathrm{d}x\right)\mathrm{d}k = \int_K \left(\int_{\Gamma\backslash G} f(xk^{-1})\overline{h(x)}\,\mathrm{d}x\right)\mathrm{d}k$$
$$= \int_K \left(\int_{\Gamma\backslash G} f(y)\overline{h(yk)}\,\mathrm{d}y\right)\mathrm{d}k = \int_{\Gamma\backslash G} f(y)\overline{\left(\int_K h(yk)\,\mathrm{d}k\right)}\,\mathrm{d}y = \int_{\Gamma\backslash G} f(y)\overline{\tilde{h}(y)}\,\mathrm{d}y = \langle f, \tilde{h}\rangle,$$

where again we used: $\int_K \mathrm{d}k = 1$, the right K-invariance of f, the Haar measure dx being right invariant and Fubini's Theorem, respectively. So f is orthogonal to h, which proves the claim. ■

Recall that $A_0 = \{\overline{(a_1, \ldots, a_d)} \in A \mid \sum a_i \equiv 0 \bmod d\}$ is the set of type-preserving translations (see Lemma 4.1). So, for any $a \in A_0$ the Hecke operator H_a is a well-defined operator from $L^2(V_i)$ to itself, for any type i.

Combining Lemmas 4.2 and 4.5, we get the following bound on the norm of the Hecke operator in terms of the matrix coefficients.

Corollary 4.6. For any type $i \in \mathbb{Z}/d\mathbb{Z}$ and any $a \in A_0$,

$$\|H_a\|_{L_0^2(V_i)} \le \sup_{f_1, f_2} \langle \rho(\pi^a) f_1, f_2 \rangle,$$

where f_1, f_2 run over all the right K-invariant normalized vectors in $L^2(\Gamma \backslash G_i)$ orthogonal to any right G^+-invariant vector (when considered as functions in $L^2(\Gamma \backslash G)$ as in Lemma 4.5.) □

Proof. Let $f_1, f_2 \in L_0^2(V_i)$ of norm 1 be such that $\|H_a\|_{L_0^2(V_i)} = \langle H_a f_1, f_2 \rangle$. By Lemma 4.2,

$$\|H_a\|_{L_0^2(V_i)} = \langle \rho(\pi^a) f_1, f_2 \rangle$$

and by Lemma 4.5, f_1, f_2 are orthogonal to any right G^+-invariant vector in $L^2(\Gamma \backslash G)$, which proves the claim. ■

4.3 Adjacency operators

Again let Γ be a co-compact lattice in G with $\Gamma \subseteq G_0$, that is, Γ preserves the type function. So $\mathcal{B}_\Gamma = \Gamma \backslash \mathcal{B}$ is a d-partite hypergraph.

Recall that for each type $i \in \mathbb{Z}/d\mathbb{Z}$, the induced bipartite graph B_i of the d-partite hypergraph $\mathcal{B}_\Gamma$ has the i-type vertices V_i on one side, and the walls E_i, that is, the simplicies of dimension $(d-2)$ of cotype i, on the other side. A vertex and a wall are connected if their union forms a maximal simplex in $\mathcal{B}_\Gamma$.

Let $\tilde{A} = \tilde{A}(B_i)$ be the normalized adjacency operator of B_i, that is,

$$\tilde{A} f(x) := \frac{1}{\deg(x)} \sum_{y \sim x} f(y),$$

where the summation is over all the neighbors of x in B_i. In the natural basis the matrix of $\tilde{A}$ is a block matrix of the form

$$\tilde{A} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix},$$

where N is a matrix of size $|V_i| \times |E_i|$.

Define D_i to be the multigraph on V_i, where two vertices are connected by as many edges as there are paths of length 2 in the graph B_i connecting them. Then the matrix NN^t is the matrix of the normalized adjacency operator of the multigraph D_i. Note that the number of loops on each vertex in D_i is equal to the vertex degree in B_i.

The nonzero eigenvalues of the matrices NN^t and N^tN coincide, and $\lambda \neq 0$ is an eigenvalue of NN^t if and only if $\sqrt{\lambda}$ is an eigenvalue of $\tilde{A}$. So, in order to bound the eigenvalues of $\tilde{A}$, it is enough to bound the eigenvalues of NN^t.

Lemma 4.7. The operator NN^t, as an operator from $L^2(V_i)$ to itself, is a convex sum of two Hecke operators $I = H_{(0,\dots,0)}$ and $H_{(-1,0,\dots,0,1)}$, in fact,

$$NN^t = \frac{1}{q+1} I + \frac{q}{q+1} H_{(-1,0,\dots,0,1)}.$$

□

Proof. By Lemma 4.1, two vertices of type i of the building share a common wall if and only if their relative position is either $\overline{(0,\dots,0)}$ or $\overline{(-1,0,\dots,0,1)}$, that is, a vertex xK of type i shares a common wall with vertices which are either the right K-cosets in $xK\pi^{(-1,0,\dots,0,1)}K$ or xK itself.

In the quotient $\mathcal{B}_\Gamma$, the vertex ΓxK of type i can be lifted to the vertex xK in the building. The i-type vertices in the building which share a common wall with xK are mapped surjectively to the vertices in the quotient which share a common wall with ΓxK in the quotient. Since Γ has injectivity radius ≥ 2, this map is also injective.

Hence, after the normalization, by the definition of the Hecke operators we get, that

$$NN^t = c_{(0,\dots,0)} H_{(0,\dots,0)} + c_{(-1,0,\dots,0,1)} H_{(-1,0,\dots,0,1)},$$

where c_a is the number of edges in D_i connecting a vertex xK to vertices of relative position a with respect to it, divided by the degree of the vertex xK in the graph D_i. Clearly $c_{(0,\dots,0)} + c_{(-1,0,\dots,0,1)} = 1$.

Each wall of the building $\mathcal{B}_d(F)$ is contained in exactly $q+1$ chambers, and each i-type vertex is contained in exactly r chambers (= facets), where the number r depends on d and q, but not on the vertex. In the quotient $\mathcal{B}_\Gamma$, since the injectivity radius ≥ 2, each wall is also contained in exactly $q+1$ chambers and each i-type vertex is contained in r chambers. Hence B_i is a bipartite $(r, q+1)$-biregular graph. Therefore, D_i is a $r(q+1)$ regular multi-graph, where each vertex has exactly r loops, so $c_{(0,\dots,0)} = \frac{1}{q+1}$, which completes the proof. ■

We can now get the following bound on the normalized second largest eigenvalue of B_i, in terms of matrix coefficients of a certain unitary representation.

Corollary 4.8. Let $(\rho, L^2(\Gamma\backslash G))$ be the unitary G-representation given by right translation $\rho(g)f(x) = f(xg)$. Let $W \leq L^2(\Gamma\backslash G)$ be the subspace of right K-invariant vectors which are orthogonal to all right G^+-invariant vectors. For any $g \in G$, define $\rho_W(g)$ to be the maximal absolute value of a matrix coefficient of normalized vectors from W on the element g, that is,

$$\rho_W(g) = \sup_{f_1, f_2 \in W, \|f_1\|=\|f_2\|=1} |\langle \rho(g) f_1, f_2 \rangle|.$$

Then the normalized second largest eigenvalue of B_i, $\tilde{\lambda}_i = \tilde{\lambda}(B_i)$ satisfies

$$\tilde{\lambda}_i \leq \sqrt{\frac{1}{q+1} + \frac{q}{q+1}\rho_W(\pi^{(-1,0,\ldots,0,1)})} \leq \sqrt{q^{-1} + \rho_W(\pi^{(-1,0,\ldots,0,1)})}.$$

□

Proof. By Lemma 4.7 and the discussion before it,

$$\tilde{\lambda}_i = \sqrt{\|NN^t\|_{L_0^2(V_i)}} \leq \sqrt{\frac{1}{q+1} + \frac{q}{q+1}\|H_{(-1,0,\ldots,0,1)}\|_{L_0^2(V_i)}},$$

and, by Corollary 4.6, $\|H_{(-1,0,\ldots,0,1)}\|_{L_0^2(V_i)} \leq \rho_W(\pi^{(-1,0,\ldots,0,1)})$. ■

4.4 Proof of the mixing Lemma

The following result by Oh gives a unified bound on the matrix coefficients of a unitary representation of a reductive group over a local field.

Theorem 4.9. [18, Theorem 1.1] Let F be a local nonarchimedean field with $char(F) \neq 2$. Let G be the group of the F-rational points of an F-split connected reductive group of rank ≥ 2 and $G/Z(G)$ almost F-simple. Let G^+ be the the subgroup of G generated by the unipotent elements of G.

Let Φ be a root system of G with regard to some maximal torus T, and Φ^+ the set of positive roots in Φ. Let $S \subset \Phi^=$ be a strongly orthogonal system of roots, which, by definition, means $\forall \alpha, \beta \in S \Rightarrow \alpha \pm \beta \notin S$.

Let K be a good maximal compact subgroup of G, which means that K is a stabilizer of a special vertex in the building of G. Any good maximal compact subgroup gives rise to a Cartan decomposition $G = K\Lambda^+K$, where Λ^+ is a positive Weyl chamber.

Then for any unitary representation ρ of G without a nonzero G^+-invariant vectors, and for any K-finite unit vectors v and u,

$$|\langle\rho(g)v, u\rangle| \leq (\dim(Kv)\dim(Ku))^{\frac{1}{2}}\xi_S(\lambda),$$

where $g = k_1\lambda k_2 \in K\Lambda^+K = G$, $\xi_S(\lambda) = \prod_{\alpha\in S}\Xi_{\mathrm{PGL}_2(F)}\left(\begin{smallmatrix}\alpha(\lambda) & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\Xi_{\mathrm{PGL}_2(F)}$ is the Harish-chandra Ξ-function of $\mathrm{PGL}_2(F)$. □

In our case, $G = \mathrm{PGL}_d(F)$ satisfies the assumptions of Theorem 4.9 and G^+ is equal to $\mathrm{PSL}_d(F)$ and every vertex is special. The subgroup K is the stabilizer of the fundamental lattice, hence K is a good maximal compact subgroup. As a maximal torus T, we take the subgroup of diagonal matrices, and as strongly orthogonal system we take the singelton $S = \{\alpha := a_d - a_1\}$, $\alpha(diag(\pi^{a_1}b_1, \dots, \pi^{a_d}b_d)) := \pi^{a_d - a_1}$, where $b_1, \dots, b_d \in \mathcal{O}^*$.

Using the following formula (see [18, Section 3.8]), for $n \in \mathbb{N}$,

$$\Xi_{\mathrm{PGL}_2(F)}\begin{pmatrix}\pi^{\pm n} & 0\\ 0 & 1\end{pmatrix} = q^{-n/2}\left(\frac{n(q-1)+q+1}{q+1}\right) \leq (n+1)q^{-n/2}.$$

We get the following application of Oh's theorem, when this time $G = \mathrm{PGL}_d(F)$.

Corollary 4.10. Define $(\rho, L^2(\Gamma\backslash G))$ to be the G-representation given by right translation $\rho(g)f(x) = f(xg)$. Let $f, f' \in L^2(\Gamma\backslash G)$ be right K-invariant unit vectors which are orthogonal to all right G^+-invariant vectors. Then for $g = \pi^{(a_1,\dots,a_d)}$,

$$|\langle\rho(g)f, f'\rangle| \leq (a_d - a_1 + 1)q^{-\frac{a_d - a_1}{2}}.$$

□

Combining all these estimates together with Corollary 3.7, we can prove the colorful mixing lemma.

Proof of the colorful mixing lemma. Let $\tilde{\lambda}_i = \tilde{\lambda}(B_i)$ be the normalized second largest eigenvalue of the bipartite graph B_i. By Corollary 4.8

$$\tilde{\lambda}_i \leq \sqrt{q^{-1} + \rho_W(\pi^{(-1,0,\dots,0,1)})}.$$

By Corollary 4.10 in the notation of Corollary 4.8 we have

$$\rho_W(\pi^{(-1,0,\dots,0,1)}) \leq 3q^{-1}.$$

Combining these together, we get that for any type i,

$$\tilde{\lambda}(B_i) \leq \frac{2}{q^{1/2}}.$$

Finally, by Corollary 3.7, for any choice of sets $W_i \subset V_i,\ i = 1, \ldots, d$,

$$\mathrm{disc}_{\Gamma\backslash\mathcal{B}}(W_1, \ldots, W_d) \le d \cdot \max_i \tilde{\lambda}(B_i) \le \frac{2d}{q^{1/2}},$$

which proves the claim. ■

5 Explicit Construction of Ramanujan Complexes

Ramanujan complexes were introduced in [9, 15, 19] as a generalization of the Ramanujan graphs constructed in [14], and were explicitly constructed in [9, 16, 19]. These complexes are certain quotients of the Bruhat–Tits buildings.

The heart of the construction in [16] is the Cartwright–Steger lattice (CS-lattice) Λ [2], which allows us to view some of the quotients of the building as Cayley complexes of finite groups with explicit sets of generators.

The reader is referred to [15, 16] for more details, and to [12] for a reader friendly survey.

5.1 The Cartwright–Steger lattice

Here we present the CS-lattice, and express explicitly its set of generators.

Let $\mathbb{F}_q$ be the finite field of size q, and $\mathbb{F}_{q^d}$ the field extension of $\mathbb{F}_q$ of degree d. Let ϕ be a generator of the Galois group $Gal(\mathbb{F}_{q^d}/\mathbb{F}_q) \cong \mathbb{Z}/d\mathbb{Z}$, and fix a basis $\xi_0, \ldots, \xi_{d-1}$ of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$ with $\xi_i = \phi^i(\xi_0)$. Denote $R_T = \mathbb{F}_q[y, \frac{1}{1+y}]$. For a given R_T-algebra S (i.e., S is given with a ring homomorphism $R_T \to S$), we define a S-algebra $\mathcal{A}(S) = \bigoplus_{i,j=0}^{d-1} S\xi_i z^j$ with the relations $z^d = 1 + y$ and $z\xi_i = \phi(\xi_i)z$. One can see that the center of $\mathcal{A}(S)$ is S, and $\mathcal{A}(S)^*/S^*$ is a group scheme for R_T-algebras.

Let $k = \mathbb{F}_q(y)$. Then $\mathcal{A}(k)$ is a k-central simple algebra. For almost all completions k_ν of k, $\mathcal{A}(k_\nu)$ splits, that is, $\mathcal{A}(k_\nu) \cong M_d(k_\nu)$. In fact, this happens for all completions except for $\nu_{\frac{1}{y}}$ and ν_{1+y}. In particular, for $F = \mathbb{F}_q((y)) = k_{\nu_y}$, the algebra splits and $\mathcal{A}(F)^*/F^* \cong \mathrm{PGL}_d(F)$ (see [16, Proposition 3.1]). On the other hand, for $\nu = \nu_{\frac{1}{y}}$ or ν_{1+y}, $\mathcal{A}(k_\nu)$ is a division algebra and $\mathcal{A}(k_\nu)^*/k_\nu^*$ is compact. Thus $\mathbb{F}_q[\frac{1}{y}, y, \frac{1}{1+y}] \hookrightarrow k_{\nu_y} \times k_{\nu_{\frac{1}{y}}} \times k_{\nu_{1+y}}$ is discrete and by substituting these rings in $\mathcal{A}(-)^*/(-)^*$ and projecting to the first coordinate $\mathcal{A}(F)^*/F^* \cong \mathrm{PGL}_d(F)$ we get a discrete subgroup, which is an arithmetic lattice.

Denote $R = \mathbb{F}_q[y, \frac{1}{y}, \frac{1}{1+y}]$. As $1 + y$ is invertible in R_T, z is invertible in $\mathcal{A}(R_T)$, since it divides $z^d = 1 + y$. Denote $b = 1 - z^{-1} \in \mathcal{A}(R_T)$. Since y is invertible in R, so is $\frac{y}{1+y}$, and hence b is invertible in $\mathcal{A}(R)$, since it divides $1 - z^{-d} = \frac{y}{1+y}$. For an element $u \in \mathbb{F}_{q^d}^* \subset \mathcal{A}(R)$, denote $b_u = ubu^{-1}$ and note that as $\mathbb{F}_q \subset R$ is in the center of $\mathcal{A}(R)$, b_u depends only on the coset $u \in \mathbb{F}_{q^d}^*/\mathbb{F}_q^*$. Define $\Sigma_1 = \{\bar{b}_u \mid u \in \mathbb{F}_{q^d}^*/\mathbb{F}_q^*\} \subset \mathcal{A}(R)^*/R^* \subset \mathcal{A}(F)^*/F^* \cong \mathrm{PGL}_d(F)$ where $F = \mathbb{F}_q((y))$. The Cartwright–Steger lattice is $\Lambda = \langle \Sigma_1 \rangle$. This is the promised CS-lattice which acts simply transitively on the vertices of the building $\mathcal{B} = \mathcal{B}_d(\mathbb{F}_q((y)))$ (see [2] and [16, Proposition 4.8]).

More explicitly, let $x_0 = [\mathcal{O}^d]$ be the vertex of the building corresponding to the standard lattice, and let $\tau : \mathcal{B} \to \mathbb{Z}/d\mathbb{Z}$ be the type function on the building. Then for each neighboring vertex x of x_0 with $\tau(x) = 1$, there exists a unique $b_u \in \Sigma_1$ such that $b_u \cdot x_0 = x$. Now, for $i = 2, \ldots, d-1$ denote $N_i = \{x \in V(\mathcal{B}) \mid x \sim x_0 \text{ and } \tau(x) = i\}$. For every $x \in N_i$, there is a unique $\gamma_x \in \Lambda$ with $\gamma_x . x_0 = x$. Let $\Sigma_i = \{\gamma_x | x \in N_i\}$, so $\Sigma_i . x_0 = N_i$. Let $\Sigma = \bigcup_{i=1}^{d-1} \Sigma_i$. Then $\Sigma . x_0$ is the set of all the neighbors of x_0, and we can identify the 1-skeleton of the building with the Cayley graph $\mathrm{Cay}(\Lambda, \Sigma)$. Note that as Λ acts simply transitively, $|\Sigma_i| = \begin{bmatrix} d \\ i \end{bmatrix}_q$ and hence $|\Sigma| = \sum_{i=1}^{d-1} \begin{bmatrix} d \\ i \end{bmatrix}_q$, where $\begin{bmatrix} d \\ i \end{bmatrix}_q$ is the number of i-dimensional subspaces of a d-dimensional vector space over $\mathbb{F}_q$.

Recall that a clique in a graph is a set of vertices such that each pair of them is connected by an edge, and the clique complex of a graph is defined to be the collection of the cliques in the graph. The clique complex of a Cayley graph is called a Cayley complex. The building $\mathcal{B}$ and its quotients with injectivity radius ≥ 2 are clique complexes, hence completely determined by their 1-skeleton.

We can conclude that if $\Gamma \lhd \Lambda$, the complex $\Gamma \backslash \mathcal{B}$ is the Cayley complex $\mathrm{Cay}(\Lambda/\Gamma, \Sigma)$ of the quotient group Λ/Γ w.r.t. the set of generators Σ. In the next subsection we will apply this in the case where Γ is a normal congruence subgroup of Λ.

5.2 Congruence subgroups

Ramanujan complexes are obtained in [16] by dividing the building modulo the action of congruence subgroups of some arithmetic co-compact lattices, such as Λ above. Here we define the congruence subgroups of Λ and display their quotients as Cayley complexes of some finite groups.

For an ideal $0 \neq I \lhd R$, define the congruence subgroup of Λ to be $\Lambda(I) = \Lambda \cap \ker(\mathcal{A}(R)^*/R^* \to \mathcal{A}(R/I)^*/(R/I)^*)$. This congruence subgroup is a finite index normal subgroup of Λ. Hence the quotient $\Lambda(I) \backslash \mathcal{B}$ is a finite simplicial complex, which we will identify with the Cayley complex of the group $\Lambda/\Lambda(I)$ (w.r.t. Σ as the set of generators). By [16, Theorem 6.2] for any $0 \neq I \lhd R$, $\Lambda/\Lambda(I)$ is a Ramanujan complex (though, in this paper, we are not really using this deep fact).

By [16, Theorem 6.6], the group $\Lambda/\Lambda(I)$ can be identified as a subgroup of $\mathrm{PGL}_d(R/I)$ which contains $\mathrm{PSL}_d(R/I)$. As $R = \mathbb{F}_q[y, \frac{1}{y}, \frac{1}{1+y}]$, we consider $I = (f)$ where $f \in \mathbb{F}_q[y]$ is an irreducible polynomial of degree $e \geq 2$. Then $R/I \cong \mathbb{F}_{q^e}$ and hence

$\mathrm{PSL}_d(\mathbb{F}_{q^e}) \leq \Lambda/\Lambda(f) \leq \mathrm{PGL}_d(\mathbb{F}_{q^e})$. Moreover, by [16, Theorem 7.1], assuming $q^e > 4d^2+1$ and $d|q^e-1$, for any subgroup $\mathrm{PSL}_d(\mathbb{F}_{q^e}) \leq G \leq \mathrm{PGL}_d(\mathbb{F}_{q^e})$, we may find f such that $G = \Lambda/\Lambda(f)$. In particular, G has a set of $\sum_{i=1}^{d-1} \begin{bmatrix} d \\ i \end{bmatrix}_q$ generators such that the corresponding Cayley complex is a Ramanujan complex.

For such quotients of $\mathcal{B}$ by congruence subgroups, a bound on the injectivity radius was presented in [13]

Proposition 5.1 ([13, Proposition 3.3]). Let $f \in \mathbb{F}_q[y]$ be an irreducible polynomial, $\Gamma = \Lambda(f)$, and $X = \Gamma\backslash\mathcal{B}$. Denote by $|X|$ the number of vertices of X. Then

$$\mathrm{dist}(\Gamma) := \min_{1\neq\gamma\in\Gamma, x\in\mathcal{B}} \mathrm{dist}(\gamma.x, x) \geq \frac{\deg(f)}{d} \geq \frac{\log_q |X|}{(d-1)(d^2-1)}. \qquad \square$$

As a consequence of this proposition, we get

Corollary 5.2 (injectivity radius). Let $X = \Gamma\backslash\mathcal{B}$ be a quotient of the building, where Γ as above. Then the injectivity radius $r(X)$ of X satisfies

$$r(X) \geq \left\lfloor \frac{\mathrm{dist}(\Gamma)-1}{2} \right\rfloor \geq \frac{\log_q |X|}{2(d-1)(d^2-1)} - \frac{1}{2}.$$

$\square$

5.3 Partite Ramanujan complexes

Before continuing, let us examine the special case of $d=2$, that is, the Ramanujan graphs (compare with [11, 14]). Assume q is an odd prime power. In this case $\Lambda/\Lambda(I)$ is a subgroup of $\mathrm{PGL}_2(\mathbb{F}_{q^e})$ containing $\mathrm{PSL}_2(\mathbb{F}_{q^e})$, and since $\mathrm{PSL}_2(\mathbb{F}_{q^e})$ is of index 2 inside $\mathrm{PGL}_2(\mathbb{F}_{q^e})$, $\Lambda/\Lambda(I)$ is either $\mathrm{PGL}_2(\mathbb{F}_{q^e})$ or $\mathrm{PSL}_2(\mathbb{F}_{q^e})$. In this case all the elements of $\Sigma = \Sigma_1$ either lie outside of $\mathrm{PSL}_2(\mathbb{F}_{q^e})$ or all are inside of it (which is the case iff the image of b is in it, in which case all the b_u, which are conjugates of b, are also there). The Cayley graph $\mathrm{Cay}(\Lambda/\Lambda(I), \Sigma)$ is bipartite in the first case and has large chromatic number in the second. In other words, the quotient $\Lambda/\Lambda(I)$ inherits the type function of the building ($\tau : \mathcal{B} \to \mathbb{Z}/2\mathbb{Z}$) if and only if the index of $\mathrm{PSL}_2(\mathbb{F}_{q^e})$ inside $\Lambda/\Lambda(I)$ is 2, that is, $\Lambda/\Lambda(I) = \mathrm{PGL}_2(\mathbb{F}_{q^e})$.

In the high-dimensional case the situation is in analogy with the 1-dimensional case (see [16, Proposition 6.7, Corollary 6.8]). Assume $d|q^e-1$ and $I=(f)$ as before with $R/I = \mathbb{F}_{q^e}$. Then $\mathrm{PGL}_d(\mathbb{F}_{q^e})/\mathrm{PSL}_d(\mathbb{F}_{q^e}) \cong \mathbb{Z}/d\mathbb{Z}$. If r is the index of $\mathrm{PSL}_d(R/I)$ in $\Lambda/\Lambda(I)$ then $r|d$, and the image of Λ in $\mathbb{Z}/d\mathbb{Z}$ under the map τ is $\frac{d}{r}\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/r\mathbb{Z}$. The quotient then inherits an r-partition from the building, that is, $\tau : \Lambda/\Lambda(I) \to \mathbb{Z}/r\mathbb{Z}$. In the construction above, the index $r=[\Lambda/\Lambda(I):\mathrm{PSL}_d(R/I)]$ is equal to the order of $\frac{y}{1+y}$ inside $(R/I)^*/(R/I)^{*d}$ (see [16, Proposition 6.7]).

Lemma 5.3. Let $0\neq I\lhd R$ and $\Gamma=\Lambda(I)$ as above, $r=[\Lambda/\Gamma:\mathrm{PSL}_d(R/I)]$, and consider the simplicial complex $\Gamma\backslash\mathcal{B}$.

(1) Denote $\Gamma_0=\Gamma\cap G_0=\{g\in\Gamma \mid \nu_F(\det(g))\equiv 0 \bmod d\}$ the subgroup of type-preserving elements of Γ. Then $[\Gamma:\Gamma_0]=\frac{d}{r}$.

(2) If $r=1$, then $\Gamma_0\backslash\mathcal{B}\to\Gamma\backslash\mathcal{B}$ is a d-cover. Moreover, for each vertex in $\Gamma\backslash\mathcal{B}$, its preimage is a set of d vertices, one of each type in $\mathbb{Z}/d\mathbb{Z}$. □

Proof. Let us look at the type function as a surjective homomorphism of groups

$$\tau=\nu_F\circ\det:\Lambda\cong\mathcal{B}^{(0)}\to\mathbb{Z}/d\mathbb{Z}.$$

If Λ/Γ is an extension of the simple group $\mathrm{PSL}_d(q^e)$ by a cyclic group of order r, it follows that by restricting the homomorphism, we get a surjective homomorphism, $\tau_\Gamma:\Gamma\to\mathbb{Z}/\frac{d}{r}\mathbb{Z}\cong r\mathbb{Z}/d\mathbb{Z}$. Now, since G_0 is the subgroup of type-preserving elements in G, then $\ker(\tau|_X)=X\cap G_0$. So by the First Isomorphism Theorem we have

$$\Gamma/\Gamma_0=\Gamma/\ker(\tau_\Gamma)\cong\tau(\Gamma)\cong\mathbb{Z}/\frac{d}{r}\mathbb{Z}.$$

This proves (a). To prove (b) we argue as follows:

Since $r=1$ then by (a) we have that $[\Gamma:\Gamma_0]=d$. Let $\gamma_1,\ldots,\gamma_d$ be representatives of Γ_0-cosets. Since $\Gamma_0=\ker(\nu_F\circ\det|_\Gamma)$ all the d types are obtained, then after renumbering, for each $i\in\mathbb{Z}/d\mathbb{Z}$, $\nu_F\circ\det(\gamma_i)=i$. Also, for any $\Gamma x\in\Gamma\backslash\mathcal{B}$, its preimages are $\Gamma_0\gamma_1x,\ldots,\Gamma_0\gamma_dx$, and their types are $1+\tau(x),\ldots,d+\tau(x)$, which give all d types in $\mathbb{Z}/d\mathbb{Z}$. ■

When $r=1$, we say that $\Gamma\backslash\mathcal{B}$ is *nonpartite*. In order to find such Ramanujan complexes we proceed as follows. Choose some $\beta\in\mathbb{F}_{q^e}^*$ such that $\beta^d\neq 1$ and that $\alpha=\frac{\beta^d}{1-\beta^d}$ generates the field $\mathbb{F}_{q^e}$. By Lemma 7.2 and the proof of Proposition 7.3 in [16], if $q^e\geq 4d^2+1$ there exists such β. Now, let $f\in\mathbb{F}_q[y]$ be the minimal polynomial of α. Then f is of degree e, and under the identification $R/(f)\cong\mathbb{F}_{q^e}$, $y\longleftrightarrow\alpha=\frac{\beta^d}{1-\beta^d}$ and $\frac{y}{1+y}\longleftrightarrow\beta^d$. Therefore $\frac{y}{1+y}\in(R/I)^{*d}$, and by the discussion above $\Lambda/\Gamma=\mathrm{PSL}_d(\mathbb{F}_{q^e})$ and the Cayley complex of Λ/Γ is nonpartite.

The above may be summarized by the following proposition, which is a special case of [16, Theorem 7.1] together with Lemma 5.3.

Proposition 5.4. Let q be a prime power, $d \geq 2$, $e \geq 2$ such that $q^e \geq 4d^2 + 1$ and $d|q^e - 1$. Let Λ be the Cartwright–Steger lattice in $\mathrm{PGL}_d(\mathbb{F}_q((y)))$. For an irreducible polynomial $f \in \mathbb{F}_q[y]$, let $\Gamma = \Lambda(f) \lhd \Lambda$ be its congruence subgroup, and let $\Gamma_0 = \Gamma \cap G_0$ be the finite index subgroup of type-preserving elements in Γ.

Then there exists an irreducible polynomial $f \in \mathbb{F}_q[y]$ of degree e, such that

(1) The Cayley complex $X = \mathrm{Cay}(\Lambda/\Gamma, \Sigma)$ is a nonpartite Ramanujan complex.
(2) The Cayley complex $\tilde{X} = \mathrm{Cay}(\Lambda/\Gamma_0, \Sigma)$ is a d-partite Ramanujan complex.
(3) The complex $\tilde{X}$ is a d-cover of X, and the preimage in $\tilde{X}$ of each vertex in X is a set of d vertices of all d types. □

Remark 5.5. A polynomial $f(y)$ of degree e can be also considered as a polynomial of degree e in $\frac{1}{y}$, by multiplying it by $\frac{1}{y^e}$, which is an invertible element in R. □

Remark 5.6. For any $d \geq 3$ and q an odd prime power such that $(d, q) = 1$, by Fermat–Euler theorem $q^{\varphi(d)} - 1 = 0 \bmod d$. Hence, there exist infinitely many integers e such that $q^e \geq 4d^2 + 1$ and $d|q^e - 1$. □

6 Proof of the Main Theorem

We are now ready to prove the following theorem which implies Theorem 1.2.

Theorem 6.1. Let $d \geq 3$ and q an odd prime power such that $(d, q) = 1$, and let X be a nonpartite Ramanujan complex as constructed in Proposition 5.4. Let $\chi(X)$ and $r(X)$ be the chromatic number and injectivity radius of X (as defined in the introduction). Then

$$r(X) \geq \frac{\log_q |X|}{2(d-1)(d^2-1)} - \frac{1}{2}$$

and, assuming $r(X) \geq 2$,

$$\chi(X) \geq \tfrac{1}{2} \cdot q^{1/2d}.$$

□

Proof. The claim about the injectivity radius is Corollary 5.2. For the chromatic number: Consider a coloring of X with $\chi(X)$ colors, and let W be the set of vertices of X of the most common color, so $|W| \geq \frac{|V(X)|}{\chi(X)}$. Let $\tilde{X}$ be the d-cover of X with the type-function inherited from building, as in Proposition 5.4. For each $i \in \mathbb{Z}/d\mathbb{Z}$, let W_i be the preimage of W of vertices of type i in $\tilde{X}$, so $|W_i| = |W|$.

Note that the image of each j-dimensional simplex $\tilde{e}$ in $\tilde{X}$ is again an j-dimensional simplex e in X. Indeed, as the injectivity radius is ≥ 2 and each simplex is a clique, so any two vertices in $\tilde{e}$ are of distance 1, hence are not mapped to the same vertex in X.

In particular, each d-dimensional simplex in $\tilde{X}$, with one vertex in each W_i, is mapped to a d-dimensional simplex in X, with all the vertices in W. But by the definition of the chromatic number there are no such simplices in X with all vertices in W, and therefore $E(W_1, \ldots, W_d) = \emptyset$.

Denote by V_i the set of vertices of type i in $\tilde{X}$, then $|V_i| = |V(X)|$ for all $i = 1, \ldots, d$. Therefore $\frac{|W_i|}{|V_i|} = \frac{|W|}{|V(X)|} \geq \frac{1}{\chi(X)}$. Since $E(W_1, \ldots, W_d) = \emptyset$, we get

$$\mathrm{disc}_{\tilde{X}}(W_1, \ldots, W_d) = \prod_{i=1}^{d} \frac{|W_i|}{|V_i|} \geq \frac{1}{\chi(X)^d}.$$

On the other hand, by the Colorful Mixing Lemma, we have

$$\mathrm{disc}_{\tilde{X}}(W_1, \ldots, W_d) \leq \frac{2d}{q^{1/2}}.$$

Combining these together we get

$$\chi(X) \geq (2d)^{-\frac{1}{d}} \cdot q^{\frac{1}{2d}} \geq \frac{1}{2} \cdot q^{\frac{1}{2d}}. \qquad \blacksquare$$

Remark 6.2. The complexes in Theorem 6.1 are nonpartite. It follows therefore that for their 1-skeletons, the largest eigenvalue of their adjacency matrices is $k \approx q^{d^2/4}$ where the second one λ_2 is at most $d^d q^{d^2/8}$ (see [12, remark 2.1.5]). It follows from [3, Theorem 1] that $\mathrm{diam}(X) \leq \frac{\log|X|}{\log(\lambda_1/\lambda_2)}$, so

$$\mathrm{diam}(X_n) \leq \frac{\log_q |X_n|}{\log_q \left(\frac{k}{d^d q^{d^2/8}}\right)} \approx \frac{\log_q |X_n|}{d^2/4 - d\log_q d} \approx \frac{4\log_q |X_n|}{d^2} \leq 8d \cdot r(X_n)$$

for $q \gg d$. So, up to a constant fraction of their diameters, these complexes are two colorable around every vertex. □

Remark 6.3. Theorems 6.1 and 1.2 are true also if either q is even or $d = 2$, but not simultaneously. In this case one should use the full power of the Ramanujan bounds. For the cases, we treated here, Oh's theorem suffices.

Note, however, that we crucially use the explicit construction of [16] where it is shown that nonpartite Ramanujan complexes do exist. This fact, in turn, depends on the work [2] of Cartwright–Steger which gives lattices acting simply transitively on the vertices of the building. The lower bound on $\chi(X)$ given in Theorem 6.1 holds for every nonpartite quotient X of $\mathcal{B}_d(F)$. □

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